EXPONENTS, LOGARITHMS, AND "e"

A Review by Ford Cochran (update by Paul A. Schroeder)

"*One day, my log will have something to say about this*." - The Log Lady

Believe it or not, exponents and logarithms are wonderful, useful things. They can describe natural processes such as radioactive growth and decay, as well as unnatural processes like the accumulation of anthropogenic (human-made) pollutants in a soil profile or the accumulation of interest in your savings account (or, perhaps more familiar, on your credit cards). You will want to understand the basic ways of using exponents and logarithms. "But wait," you say, "isn't this why they made accountants?" Indeed it is. But with a little practice, you will find working with "e" and "ln" so easy that you will scoff at the notion of subcontracting. What's more, it may come in handy on the mid-term.

First, if you don't already own a scientific or business calculator, borrow one or start the scientific calculator on your phone. The other possibility is to borrow a really expensive calculator known as a computer spread sheet (for example, Excel). Do this now, before you read another word. Good job. If you haven't used the calculator's exponential functions before, find the keys marked "log" and " 10^X" or "alog," as well as those marked "ln" and "exp" or "e^X." Turn the thing on.

If you are using a spreadsheet, then you'd have start the program and open a worksheet. In all the spreadsheet examples below we will assume the number you want to "operate" on, is in the first cell A1. The calculations will be conducted in cell B1.

You are probably familiar with the powers of 10:

> $10^0 = 1$ (By convention, any number to the zero power is one.)
> $10^1 = 10$ (Any number to the first power is itself.)
> $10^2 = 10$ squared $= 10 \times 10 = 100$
> $10^3 = 10$ cubed $= 10 \times 10 \times 10 = 1000$

If you have the number 10 in cell A1, then calculate powers of 10 in cell B1 with the following syntax:

> **=A1^0**
> **=A1^1**
> **=A1^2**
> **=A1^3**

And so on. You can also raise 10 to negative powers, which may seem like an odd sort of thing to do until you recognize that 10^{-2} simply means $1/10^2$.

> $0^{-1} = 10$ to the negative first $= 1 / 10 = 0.1$
> $10^{-2} = 10$ to the negative second $= 1/ (10 \times 10) = 0.01$
> $3.76 \times 10^{-3} = 3.76/ (10 \times 10 \times 10) = 3.76 \times 0.001 = 0.00376$

Likewise for the spreadsheet, using the number 10 in cell A1, typing **=A1-2** in cell B1 would produce an answer of 0.01.

Still more miraculous, you can raise a number to a fractional or decimal power. $10^{0.6}$, for instance, is 3.98107.... If you don't believe me, try this on your calculator: Enter 0.6, then hit the "10^X" or "alog" key (they both do the same thing) and see what you get.

In your spreadsheet, you can see that using the number 10 in cell A1, typing **=A1-.6** in cell B1 would produce an answer of 3.98107….

You can see that because of the way exponents are defined, 10 to a large negative number (minus 50 is large enough) is about zero, 10 to the zero is one, and l0 to a very large positive number (say plus 50) is a huge positive number. 10 to any power are always positive, even if that power is negative.

Why, you may be wondering, should "alog" mean the same thing as 10 to the x? Well, it's like this: alog is short for anti-logarithm (base 10 is understood) just as log is short for logarithm (base 10). The base 10 logarithm of a number is the power to which 10 must be raised to get that number. Thus log 100 or "the base 10 logarithm of 100" is simply 2, since 10 squared or 10^2 is 100. Similarly, log 0.001 is -3 and log (3.98107) is 0.6. It's no surprise, then, that alog or antilog is simply the inverse of log: If log 100 equals 2, then it is only right and just that alog 2 should be 10^2 or 100. And it is, as your calculator will verify. Oh brave new world, that has such functions in it! To restate:

If log (x) = y, then 10^y or alog (y) = x; if 10^x or alog (x) = y, then log (y) = x log (1000) = 3 means 10^3 = 1000, and vice versa

In your spreadsheet, you can easily find the log of a number. For example, by having 10 in cell A1 and typing **= log (A1)** in cell B1, you would produce an answer of 1.

Now that you've mastered base 10 logarithms and exponents, you're ready to move up to e. What, exactly, is e? It's simply a number: 2.71828.... The trailing dots indicate that, like π, it is an irrational number, which means you can keep going as long as you like but you'll never get to the end of it. Fortunately, you don't have to. That's the calculator's/computer's problem. But why, you may complain, would I want to trade a number like 10 - so round, so satisfying - for e? The reason is that for many problems (such as those you'll be solving), e is an easier and more natural base to work in than base 10. Hence, and for other technical reasons, the number e is also called "the base of natural logarithms."

In terms of doing calculations, e works just like 10. e squared or e^2 is simply e times e, and e^{-4} is 1 / e^4 or 1 / (e times e times e times e). The only thing that might make base e seem more difficult than base 10 is that, unless you're an inveterate math type, you can't square or cube e in your head. (I know I can't!) But that's no problem - we have calculators and computers to do it for us. So to find out what e^2 is, you just punch the number 2 into your calculator and hit "e^x" or "exp" (which means the same thing) and calculator tells you it's 7.389… How will you find out what e equals if you forget? Easy. A number to the power 1 is always the same number, so enter 1 on your calculator and hit e^x, and the calculator gives you the value of e. In your spreadsheet, you can easily find e^2. By having 2 in cell A1 and typing **= exp (A1)** in cell B1, you would produce an answer of 7.389….

Reversing the process, if you type the number e into your calculator and hit the "ln" button, the calculator returns the number 1. Just as log stands for "base 10 logarithm," so ln stands for "base e logarithm," or "the power to which I have to raise e to get this number." ln (1) is 0, since e to the 0 equals 1. ln (7.389) is about 2, since e to the 2 or (e x e) equals roughly 7.389, with a little round-off error. In your spreadsheet, you can find ln(7.389). By having 7.389 in cell A1 and typing **= ln (A1)** in cell B1, you would produce an answer of 1.99999…. or by rounding off, the number 2.

Like powers of 10, negative powers of e run from nearly 0 to nearly 1, e to the 0 is 1, and positive powers of e run from just over 1 to infinity. Note that since e to any power is positive, taking ln of a negative number is meaningless. See what happens when you ask your calculator for the base e logarithm of -1. So we have:

If In (x) = y, then $e^y = x$

If e^x or exp (x) = y, then ln (y) = x

ln (7.389) = 2, means $e^2 = 7.389$, and vice versa

Working with exponents and logarithms is in many ways easier than working with ordinary numbers. For instance:

(1) $10^a \times 10^b = 10^{a+b}$ and $e^a \times e^b = e^{a+b}$

This only makes sense: $10^2 \times 10^3 = (10 \times 10) \times (10 \times 10 \times 10) = 10^5 = 10^{2+3}$

(2) $10^a / 10^b = 10^{a-b}$; $e^a / e^b = e^{a-b}$

Again, sensible: $10^2/ 10^3 = (10 \times 10) / (10 \times 10 \times 10) = 10^{-1} = 10^{2-3}$

(3) $(10^a)^b = 10^{a \times b}$; $(e^a)^b = e^{a \times b}$

For instance, $(10^2)^3 = (10 \times 10)^3 = (10 \times 10) \times (10 \times 10) \times (10 \times 10) = 10^6 = 10^{2 \times 3}$

The analogous rules for the manipulation of logarithms is:

(1') log a + log b = log (a x b)

log (100) + log (1000) = 2 + 3 = 5 = log (100,000) = log (100 x 1000)

(2') log a - log b = log (a / b)

log (100) - log (1000) = 2 - 3 = -1 = log (0.1) = log (100/ 1000)

(3') log (a^b)= b x log a

$\log(10^3) = \log(1000) = 3 = 3 \times 1 = 3 \times \log(10)$

As a special case of (3'), $\log(c / d) = \log[(d / c)^{-1}] = -\log(d / c)$ $\log(0.01) = \log(100^{-1}) = -\log(100) = -2$

Replace "log" by "ln" above and the same general rules hold for base e logarithms. I'm just too lazy to type them out.

Now, perhaps the most important and astonishing property of all, because log x and 10^x are inverse functions, they undo each other:

$$\log(10^a) = a$$

$$10^{\log(b)} = b$$

Ditto for base e:

$$\ln(e^a) = a$$

$$e^{\ln(b)} = b$$

It's like wrapping and unwrapping a present: e wraps up the a but "ln" unwraps it, "ln" wraps up the b but e unwraps it. You will find this simple fact useful in your work. You may also find it aesthetically satisfying. As an example of how all this might prove useful, imagine you have an equality with e to some power on one side of it. You can whack both sides of the equality with "ln" to vaporize the e. Conversely, if the "ln" on one side of an equality is causing unrest, you can raise e to the power of each side and the distressing "ln" vanishes. More concretely, take a familiar relation from radioactive decay. (Don't panic. I'm just using this as an example and it will be explained thoroughly in due course.) N is the amount of parent isotope remaining after t years, N_o is the amount you started with, λ is the isotope's decay constant, and t is the elapsed time. Then:

$$N = N_o e^{-\lambda t}$$

Dividing by N_o: $(N / N_o) = e^{-\lambda t}$

Slapping both sides with "ln": $\ln(N / N_o) = \ln(e^{-\lambda t}) = -\lambda t$

Dividing by $-\lambda$ gives us: Elapsed time $= t = [\ln(N / N_o)] / (-\lambda)$

This means that if we know how much of a parent isotope we started with in, let us suppose, a rock, and how much of the isotope we have left today, and the rate at which the isotope decays, we can calculate the age of the rock. Swell!

Before some practice problems, a quick note on significant digits. All sorts of scientific problems involve approximations, compromises, and irrational numbers like e. The numbers we calculate with our data cannot be more exact than the data itself. Suppose I want to find the area of my desk top, which as you know is its length times its width. If I have a ruler with millimeter divisions on it, I might measure and learn that my desk is 115.3 centimeters long by 78.7 centimeters wide. Multiplying these values on my calculator, I learn that my desk is 9074.11 square centimeters in area. Or do I? This is certainly what my calculator tells me. But in fact, my calculator has assumed that the dimensions of my desk are 115.300 cm by 78.7000 centimeters, since the product of two numbers cannot have more significant (i.e., accurate, meaningful) digits in it than the original numbers themselves. Since my ruler only gives me three significant digits on the width (it doesn't measure fractions of millimeters), I only really know that the area of my desk is about 9070 square centimeters. It might be 9071 or 9077. In short, the calculator has given me some superfluous digits. Similarly, 356 meters (plus or minus a meter error) added to 2.126 meters (plus or minus a millimeter, or .001 meter) equals 358 meters, not 358.126 meters. Why? Because the error in the first measurement swamps the last three digits in the second, rendering them meaningless in the final sum. People often report every number their calculator gives them, which would be equivalent to my taking an ordinary ruler to my desk and then reporting its area to 12 decimal places. While this may lift your Boss's, or Professor's or TA's spirits, the reward it gets you may cause yours to sink! Scientific notation makes it easy to see how many significant digits you have in your answer: the number 50,000 might reflect from one to five significant digits, but 5.00×10^4 has just three.

Finally, some practice problems, not to be turned in, but good for the soul. Be sure you can get numerical as well as conceptual answers. If you have trouble solving these with a calculator or they don't make sense to you, speak to your Boss, TA or Professor:

a) $\log 10 + \log 3 =$

b) $\log 30 =$

c) $\log 7.5 =$

d) $\log 10^{3.3} =$

e) $\log (1.5 \times 10^{2.1}) =$

f) $10^{-3}=$

g) $e^{-6.894} =$

h) $\ln (e^{-572}) =$

i) $e^{\ln [\ln (e)]} =$